

Calculation of the Space-Time Curvature in the Vicinity of Schwarzschild Black Holes: Application to Sagittarius A*

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Abstract

We have calculated values of space-time curvature in the vicinity of Schwarzschild black holes. For this we have used the metric and the results of Schwarzschild collected in his study on relativistic gravitation published in 1916 as well as techniques of differential geometry of surfaces. This has allowed us to obtain values of space-time curvature in the vicinity of SAGITTARIUS A*, and thereby giving a physical content to our mathematical approach to the subject. The results are consistent with what was expected.

Keywords: Black holes; Curvature of space-time; Cosmology; General relativity

Introduction

The physical problem that concerns us is the calculation of the curvature of space-time caused by a spherical black hole at a point located at a distance "r" from the center of the black hole. This point will always be at a greater distance from the event horizon or Schwarzschild radius, " R_s " also called this distance radius of the black hole. Schwarzschild solves the equations of the theory of generalized relativity for a supposition of a spherical mass and a surrounding empty space, establishing a metric and an equation of space-time that turns out to be the Flamm paraboloid [1]. This approach leads to a stationary space-time solution in time and the spherical symmetry of the problem geometrically simplifies its solution, resulting in a 2D surface, as represented in **FIG 1**.

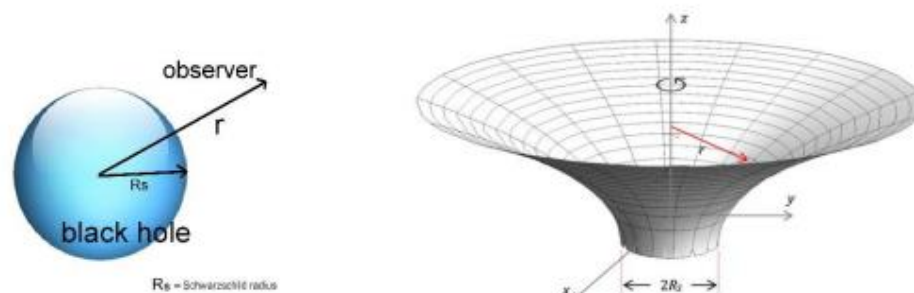


FIG. 1 The physical problem in Euclidean space.

Space-time in the Schwarzschild metric Flamm paraboloid.

Resolution of the Mathematical Problem

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Flamm's paraboloid, mathematical solution to the proposed physical problem, is a surface inserted in a space R^3 . Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the center of the black hole “r” and the azimuth angle “φ”. The problem admits a mathematical treatment of differential geometry of surfaces, and with it we are going to calculate values of Gaussian Curvature and Mean Curvature [2],

Surface parameters (r, φ)

$$0 \leq r < \infty, 0 \leq \varphi < 2\pi$$

which has this parametric vector equation:

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \\ z &= 2(R_s(r - R_s))^{1/2} \end{aligned}$$

and by vector equation:

$$f(x, y, z) = (r \cos \varphi, r \sin \varphi, 2((R_s(r - R_s))^{1/2})$$

Determination of velocity, acceleration, and normal vectors to the surface,

$$\begin{aligned} \partial f / \partial \varphi &= (-r \sin \varphi, r \cos \varphi, 0) & \partial^2 f / \partial \varphi^2 &= (-r \cos \varphi, -r \sin \varphi, 0) \\ \partial f / \partial r &= \left(\cos \varphi, \sin \varphi, (r / R_s - 1)^{-1/2} \right) & \partial^2 f / \partial r^2 &= \left(0, 0, (-1 / (2R_s)) \times (r / R_s - 1)^{-3/2} \right) \\ \partial f / \partial \varphi \partial r &= (-\sin \varphi, \cos \varphi, 0) \\ n &= \left(\partial f / \partial \varphi \times \partial f / \partial r \right) = (r \cos \varphi / (r / R_s - 1)^{1/2}, r \sin \varphi / (r / R_s - 1)^{1/2}, -r) \\ [n] &= r \left(\left(1 / (r / R_s - 1) \right) + 1 \right)^{1/2} \\ n &= n / [n] \end{aligned}$$

Curvature and curvature parameters,

Gauss curvature: $K=LN-M^2/EG-F^2$

Mean curvature: $H=GL-2FM+EN/2(EG-F^2)$

$$L = \partial^2 f / \partial \varphi^2 \cdot n$$

$$N = \partial^2 f / \partial r^2 \cdot n$$

$$M = (\partial f / \partial \varphi \partial r) \cdot n$$

$$E = \partial f / \partial \varphi \cdot \partial f / \partial \varphi$$

$$G = \partial f / \partial r \cdot \partial f / \partial r$$

$$F = \partial f / \partial \varphi \cdot \partial f / \partial r$$

We have particularized the equations in 12 points between 1 and 600 Schwarzschild radii calculating the corresponding curvatures as shown in the results **TABLE 1**. Thus, although in the metric there is a singularity at the point $1R_s$, in the value of the Gaussian curvature and mean curvature the singularity is resolved mathematically calculating a limit. We have calculated that limit for Gauss curvature and Mean curvature.

Results of Curvature Values

According to the exposed mathematical approach, we have calculated curvature values at 12 points located at a distance between 1 and 600 Schwarzschild radii (R_s).

TABLE 1. Calculated curvature values.

	Distance to the center of the black hole	Value of Gauss Curvature (k)	Value of Mean Curvature (H)
	$1R_s$	$-0,5000 \times R_s^{-2}$	$-0,2500 \times R_s^{-1}$
<i>C</i>	$1,2R_s$	$-0,2873 \times R_s^{-2}$	$-0,1888 \times R_s^{-1}$
<i>U</i>	$1,4R_s$	$-0,1821 \times R_s^{-2}$	$-0,1509 \times R_s^{-1}$
<i>R</i>	$1,6R_s$	$0,1220 \times R_s^{-2}$	$-0,1236 \times R_s^{-1}$
<i>V</i>	$1,8 R_s$	$-0,0790 \times R_s^{-2}$	$-0,1119 \times R_s^{-1}$
<i>A</i>	$2R_s$	$-0,0625 \times R_s^{-2}$	$-0,0543 \times R_s^{-1}$
<i>T</i>	$3R_s$	$-0,0186 \times R_s^{-2}$	$-0,0485 \times R_s^{-1}$
<i>U</i>	$4R_s$	$-0,0078 \times R_s^{-2}$	$-0,0312 \times R_s^{-1}$
<i>R</i>	$5R_s$	$-0,0030 \times R_s^{-2}$	$-0,0292 \times R_s^{-1}$
<i>E</i>	$6R_s$	$-0,0023 \times R_s^{-2}$	$-0,0168 \times R_s^{-1}$
<i>S</i>	$60R_s$	$-2,3 \cdot 10^{-6} \times R_s^{-2}$	$-5,3 \cdot 10^{-4} \times R_s^{-1}$
	$600R_s$	$-2,8 \cdot 10^{-9} \times R_s^{-2}$	$-2,3 \cdot 10^{-5} \times R_s^{-1}$

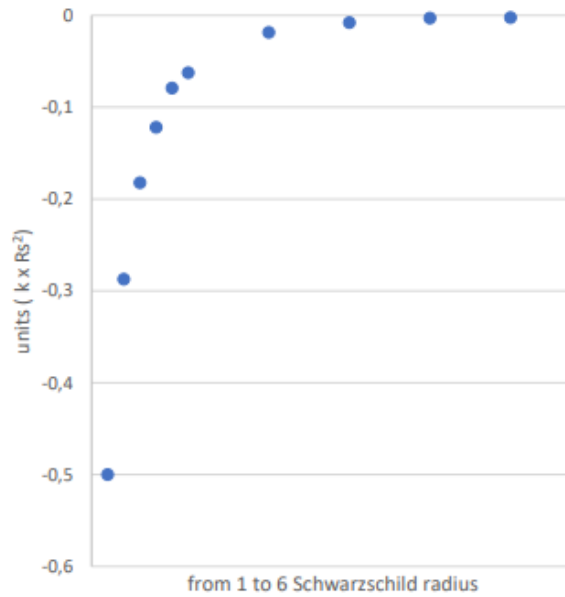


FIG. 2 Gauss curvature of space-time.

We have analyzed in more detail the width between 1 and 2 Schwarzschild radius, determining 5 points of curvature with at distance intervals of 0, 2 Schwarzschild radii. We have also calculated five points of curvature between 2 and 6 Schwarzschild radii, at distance intervals of one Schwarzschild radius. Lastly, we have calculated the curvature at two more distant points corresponding to 60 and 600 Schwarzschild radii.

Application to the calculation of the curvature in the vicinity of SAGITTARIUS A*

Sagittarius A* is the supermassive black hole at the galactic center of the Milky Way. Like the nuclei of most spiral and elliptical galaxies, the Milky Way contains a black hole at its center. The mass of SAGITTARIUS A* is estimated at 4 million suns,

approximately 3.4×10^{36} Kg, its Schwarzschild radius, R_s , thus turns out to be 5×10^9 meters [3]. According to these data, the space-time curvature values in its vicinity turn out to be (TABLE 2):

TABLE 2 The space-time curvature values in its vicinity.

Distance to the center of the black hole	Value of Gauss Curvature (k)	Value of Mean Curvature (H)
$1R_s$	$-2,0 \times 10^{-20} \text{ m}^{-2}$	$-0,5 \times 10^{-10} \text{ m}^{-1}$
$1,2R_s$	$-1,1 \times 10^{-20} \text{ m}^{-2}$	$-0,4 \times 10^{-10} \text{ m}^{-1}$
$1,4R_s$	$-0,7 \times 10^{-20} \text{ m}^{-2}$	$-0,3 \times 10^{-10} \text{ m}^{-1}$
$1,6R_s$	$-0,5 \times 10^{-20} \text{ m}^{-2}$	$-0,2 \times 10^{-10} \text{ m}^{-1}$
$1,8 R_s$	$-0,3 \times 10^{-20} \text{ m}^{-2}$	$-0,2 \times 10^{-10} \text{ m}^{-1}$
$2R_s$	$-0,2 \times 10^{-20} \text{ m}^{-2}$	$-0,1 \times 10^{-10} \text{ m}^{-1}$
$3R_s$	$-0,7 \times 10^{-21} \text{ m}^{-2}$	$-0,9 \times 10^{-11} \text{ m}^{-1}$
$4R_s$	$-0,3 \times 10^{-21} \text{ m}^{-2}$	$-0,6 \times 10^{-11} \text{ m}^{-1}$
$5R_s$	$-0,1 \times 10^{-21} \text{ m}^{-2}$	$-0,5 \times 10^{-11} \text{ m}^{-1}$
$6R_s$	$-0,9 \times 10^{-22} \text{ m}^{-2}$	$-0,3 \times 10^{-11} \text{ m}^{-1}$
$60R_s$	$-9,2 \times 10^{-26} \text{ m}^{-2}$	$-1,0 \times 10^{-13} \text{ m}^{-1}$
$600R_s$	$-1,1 \times 10^{-28} \text{ m}^{-2}$	$-4,6 \times 10^{-15} \text{ m}^{-1}$

Analysis of Results

We have obtained the results of the curvatures through a differential geometry approach, specifying the calculation in 12 points between 1 and 600 Schwarzschild radii. The calculation has been very laborious, and the results have been checked several times to verify their accuracy.

The curvature values are negative; this may imply a physical approach in the vicinity of the black hole to be considered in the future. The second is the finite value of the curvature at the event horizon or radius of the black hole, $1R_s$. Although the Schwarzschild metric has a singularity in point $1R_s$, there, this is mathematically avoided in the calculation of the curvature, resulting in a finite value of it. It should be noted that in the non-relativistic case, that is, for Newton's gravity, the curvature would be proportional to $1/r^2$, deviating noticeably from this assumption such as can be seen in FIG. 2 [4].

Using these results, we have calculated the curvature of space-time in the region close to a Schwarzschild black hole, this has allowed us to obtain curvature values in the vicinity of SAGITARIUS A*, and thus give a physical result to our mathematical calculations.

An Equation for Calculating Curvatures in the Vicinity of Schwarzschild Black Holes

We are going to study the Gaussian curvature data by means of an equation obtained by fitting.

Using an Excel program to determine the regression equation. We have not been able to obtain a function that fits all the data with a high degree of accuracy. For this reason, we have decided to fit only the data that are at distances from the center of the black hole between 1 and 7 Schwarzschild radii and for which we have found an equation that fits the processed data with a degree of accuracy of 99.98%. The result of the fit is the following:

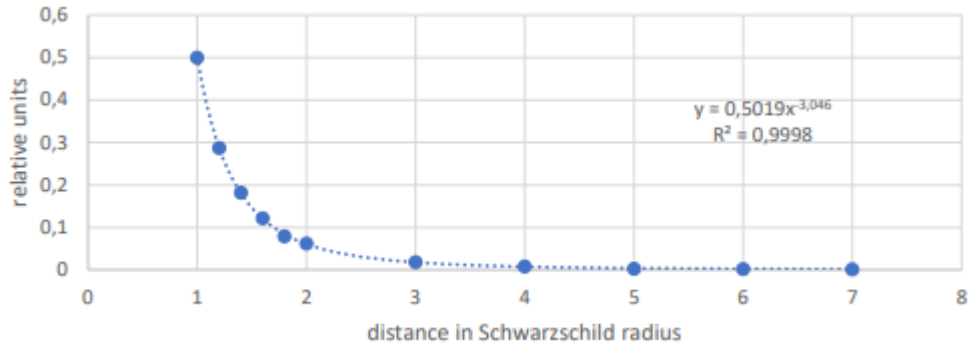


FIG. 3 Comparison between the curvature values obtained by the Schwarzschild Model and those obtained by the fit equation.

Fit equation between 1 and 7 Schwarzschild radii:

Gaussian curvature	$k = -0,5019 \left(r / R_s \right)^{-3,046} \times R_s^{-2}$
Fit quality	$R^2 = 0,9998$

We express our formula as follows:

$$k = -0,5 R_s / r^3$$

$$R_s = 2GM / c^2$$

$$k = -0,5 \left(2GM / c^2 r^3 \right)$$

$$k = -GM / c^2 r^3$$

We are going to test our formula by calculating curvature values corresponding to the black hole SAGITTARIUS A* and comparing them with those obtained by us in this study according to the Schwarzschild model.

TABLE 3 Results obtained using the equation and the results obtained from the Schwarzschild model.

Mass of SAGITTARIUS A* =3,4×10 ³⁶ Kg	G = 6,67×10 ⁻¹¹ N.m ² /kg ²	k = -GM/c ² r ³	k = -2,52.10 ⁹ /r ³
Comparison between curvature values for SAGITTARIUS A*			
Distance in Schwarzschild radius	Distance in meters	k = - 2,52×10 ⁹ /r ³	k (Values according to the Schwarzschild model)
1	5.10 ⁹	-2,0.10 ⁻²⁰ m ⁻²	-2,0.10 ⁻²⁰ m ⁻²
1,2	6.10 ⁹	-1,1.10 ⁻²⁰ m ⁻²	-1,1.10 ⁻²⁰ m ⁻²
1,4	7.10 ⁹	-0,7.10 ⁻²⁰ m ⁻²	-0,7.10 ⁻²⁰ m ⁻²
1,6	8.10 ⁹	-0,5.10 ⁻²⁰ m ⁻²	-0,5.10 ⁻²⁰ m ⁻²
1,8	9.10 ⁹	-0,3.10 ⁻²⁰ m ⁻²	-0,3.10 ⁻²⁰ m ⁻²
2	10.10 ⁹	-0,2.10 ⁻²⁰ m ⁻²	-0,2.10 ⁻²⁰ m ⁻²
3	15.10 ⁹	-0,7.10 ⁻²¹ m ⁻²	-0,7.10 ⁻²¹ m ⁻²
4	20.10 ⁹	-0,3.10 ⁻²¹ m ⁻²	-0,3.10 ⁻²¹ m ⁻²
5	25.10 ⁹	-0,1.10 ⁻²¹ m ⁻²	-0,1.10 ⁻²¹ m ⁻²
6	30.10 ⁹	-0,9.10 ⁻²² m ⁻²	-0,9.10 ⁻²² m ⁻²

7	$35 \cdot 10^9$	$-0,5 \cdot 10^{-22} \text{ m}^{-2}$	$0,6 \cdot 10^{-22} \text{ m}^{-2}$
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As shown in the **TABLE 3**, the results obtained using the equation and the results obtained from the Schwarzschild model are similar, deviating slightly only in the value of the curvature at the point corresponding to a distance from the center of the black hole of 7 Schwarzschild radii.

For this reason, we consider that the formula obtained is highly efficient in calculating curvatures of space-time in the vicinity of Schwarzschild black holes in the range of distances to the center of the black hole between 1 and 7 Schwarzschild radii.

Conclusion

Using the results of the Schwarzschild solution to Einstein's generalized relativity equations; we have calculated some values of space-time curvature in the vicinity of regular black holes. We have applied these results to the black hole of our galaxy SAGITTARIUS A*, calculating the curvature of space-time in its vicinity. Finally, we have obtained a formula that allows us to calculate curvature values at distances from the center of the black hole between 1 and 7 Schwarzschild radii obtaining excellent results with it when we have tested it for the case of SAGITTARIUS A*.

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