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A Better Solution the Precession of Mercury's Perihelion

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Abstract

The article shows that the explanation that Einstein gave to the precession of the perihelion of Mercury is incorrect: the dynamic equations he used do not even accelerate a falling stone, they cannot be used as an improvement of Newtonian mechanics. Then the article derives a formula for the precession speed and shows why most of the precession of Mercury can be explained by gravitational forces from other planets. But these forces change in time, the last section calculates a long time average of the effect of Jupiter on Mercury's precession speed. This effect is about one hundred times smaller than the relatively short term effect that has been measured. This means that actually Mercury's long term precession is much smaller than it seems to us based on our relatively short time series when the precession has been measured. This long term precession effect is quite on the range of the unexplained small part of Mercury's precession and it might be a mechanism that has not been considered. The last section shows a serious error in the relativistic calculation of the precession speed of Mercury.

Keywords: Mercury; Gravitational forces; Speed; Relativistic; Calculation

Introduction

The precession of Mercury's perihelion has been measured to 5600 arcseconds in a century. Of this figure known mechanisms can explain at most 5557 archseconds when the error bounds of the estimated precession for each mechanisms is taken to the maximum limit. Still 43 archseconds in a century remain unexplained and there must exist some unknown or overlooked mechanism or mechanisms. Einstein gave a formula derived from the General Relativity Theory. This formula gives exactly 43 archseconds, which is rather surprising as it means that all known mechanisms did reach the maximum error limits. A figure that is a bit higher than 43 archseconds in a century would be more believable. Einstein's formula also predicts very well the precession of the perihelion of Venus, but it is not equally accurate in the precession speed of the Earth. There is no known reason why the formula would be less accurate in some cases.

Einstein used in his calculation a dynamic equation derived from a geodesics of the Schwarzschild metric. The first section proves that this approach cannot be used to calculate corrections to Newtonian gravitational theory because the same method that Einstein used for Mercury gives a dynamic equation for a stone falling from the Pisa tower. A stone falling according to Einstein's dynamic equation does not accelerate at all. As the method fails to explain the old Pisa stone dropping experiment, which Newtonian gravity

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quite correctly explains for all practical purposes, it cannot be considered as valid method for calculating fine corrections to Mercury's orbit. Einstein's formula must be seen as heuristic: it gives good results in some cases (there are only few planets and moons), but lacks a sound theoretical basis.

The second section of the presented article derives an equation for the precession speed and shows with a simple model that the equation fits well to the gravitational effect of Jupiter in the rather short time period when the precession of Mercury has been measured. The whole precession cycle is over 23,000 years, therefore full precession cycles have never been measured scientifically.

The third section calculates a long term gravitational effect of a planet on the precession of Mercury. The result shows that Jupiter's long term effect on the precession speed of Mercury is about one hundred times smaller than Jupiter's effect on the relative short time period when Mercury's precession has been measured. The long term effect is about 54 archseconds in a century and such long term effects may explain the missing 43 archseconds in a century, a value that more likely is a bit bigger than 43.

The fourth section looks at the way Einstein's formula for precession is derived. The section shows that the Lagrangian is incorrectly calculated, $\mathcal{L}$ is not constant. This invalidates the calculation of precession speed. Then the section shows that the curve that Einstein's Lagrangian gives is not a rotating ellipse and that it gives an impossible relation for the impulse momentum. In short, the geodesic Lagrangean is completely wrong and useless.

The Error in Einstein's Calculation

In General Relativity dynamic equations of a test mass are Euler-Lagrange equations calculated from a geodesic Lagrangean

$$\mathcal{L} = \sqrt{g_{ab}\dot{x}^a\dot{x}^b} \qquad \text{where} \qquad \dot{x}^a = \frac{d}{d\tau}x^a \tag{1}$$

and τ is the proper time. The Lagrangian is chose to have the value L = 1 as it simplifies calculating the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial x^a} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^a} = 0. \tag{2}$$

In the calculation of the precession of the perihelion of Mercury Einstein derived the equation of motion from a geodesic in the Schwarzschild metric, probably because the gravitational field must approximate the Newtonian gravitational field around the Sun. The field that the Sun creates seems to be time-independent and spherically symmetric at least to some rather high degree of precision. The only time-independent and spherically symmetic solution to the Einstein equations that can be considered as approximating Newtonian gravity in some sense is the Schwarzschild metric.

The Schwarzschild metric is defined as

$$c^2d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2(\theta)d\phi^2 \tag{3}$$

where

$$A(r) = c^2\left(1 - \frac{r_s}{r}\right) \qquad B(r) = \left(1 - \frac{r_s}{r}\right)^{-1} \tag{4}$$

and r_s is a constant called Schwarzschild radius. This metric describes the gravitational field created by a mass at the origin. We will denote this mass by M.

Let us find the equation of motion for a test mass *m* falling straight to the mass center at the origin. This means that $\dot{\phi} = 0$ and $\dot{\theta} = 0$. The Lagrangean is

$$\mathcal{L} = \sqrt{A(r)\dot{t}^2 - B(r)\dot{r}^2}. \tag{5}$$

We get Euler-Lagrange equations only for *t* and for *r*. For *t*

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \tag{6}$$

as the field is time-independent, while

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{d}{d\tau}\left(2A(r)\dot{t}\right)(2\mathcal{L})^{-1} = 0. \tag{7}$$

Notice how nice it is that $L = 1$, the division with the square root is division with one.

The equation (7) implies that $A(r)\dot{t} = C_1$, a constant. As $A(r) = B(r)^{-1}$ in (4)

$$B(r) = \frac{\dot{t}}{C_1}. \tag{8}$$

Taking the partial derivative with respect to *r* from (8) gives

$$\frac{\partial}{\partial r}B(r) = \frac{\partial}{\partial r}\frac{\dot{t}}{C_1} = 0. \tag{9}$$

but as *B(r)* is only a function of *r*,

$$0 = \frac{\partial}{\partial r}B(r) = \frac{d}{dr}B(r) = B'(r). \tag{10}$$

Thus, $B(r) = C_2$, a constant. This observation does not agree with (4), but we pretend not to know what *B(r)* is, let us continue. Then $A(r) = B(r)^{-1} = C_2^{-1}$ is also a constant and

$$\dot{t} = C_1 C_2^{-1}$$

shows that

$$t = C_1 C_2^{-1}\tau + C_3 \tag{11}$$

where C_3 is yet another constant. Calculating the Euler-Lagrange equation for *r* we get

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial}{\partial r}(A(r)\dot{t}^2 - B(r)\dot{r}^2)(2\mathcal{L})^{-1} = \frac{\partial}{\partial r}(C_2^{-1}\dot{t}^2 - C_2\dot{r}^2)2^{-1} = 0 \tag{12}$$

and therefore

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{d}{d\tau}(2B(r)\dot{r})(2\mathcal{L})^{-1} = C_2\frac{d}{d\tau}\dot{r} = 0 \tag{13}$$

where we used $L = 1$. Thus

$$r = C_4\tau + C_5 \tag{14}$$

for some constants C_4 and C_5. Proper times cannot be directly observed, but we can observe

$$\frac{d}{dt}r = \frac{d\tau}{dt}\frac{dr}{d\tau} = C_1C_2^{-1}c_4. \tag{15}$$

That is a linear equation, thus

$$\frac{d^2}{dt^2}r = 0. \tag{16}$$

The stone does not accelerate while freely falling in a gravitational field.

This is not the only problem in the Schwarzschild metric and General Relativity. The Schwarzschild metric is not a valid metric at all: writing it in local Cartesian coordinates there are cross terms $dx_i dx_j, i \neq j$. Such cross terms cannot appear in any Riemannian metric with orthogonal coordinates and Cartesian local coordinates are orthogonal. The Schwarzschild metric does not converge to a Minkowski metric when the local environment shrinks. This is fatal: when the local environment is made smaller, curvature of the space decreases. The tangent space is flat and it should be a Minkowski space. It is not for the Schwarzschild metric. This is the reason why the speed of light is not constant in the Schwarzschild metric. In the Schwarzschild metric the speed of light sent horizontally has a speed that depends on the altitude, it would be measurable. The Einstein equations do not allow any spherically symmetric solution that has locally constant speed of light in vacuum. For proofs of these statements see [1-5].

Deriving a Formula for the Precession Speed

An ellipse is defined by

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{17}$$

where a and b are semi-major and semi-minor axes, $a \geq b > 0$. The focus points are $(-c, 0)$ and $(c, 0), c \geq 0$, and in this article the rotation center is at $(-c, 0)$. Eccentricity is defined as $e = c/a$. Notice that $b^2 = a^2 = c^2$. Coordinates (x, y) are centered at origin. Polar coordinates (r_1, ϕ) are centered at (−c, 0), thus

$$r_1 = \sqrt{(x+c)^2 + y^2} = ex + a \tag{18}$$

$$r_1\cos(\phi) = x + c \qquad r_1\sin(\phi) = y. \tag{19}$$

Solving r_1 from (28) and (29)

$$r_1 = e(r_1\cos(\phi) - c) + a \tag{20}$$

gives

$$r_1 = a\frac{1-e^2}{1-e\cos(\phi)}. \tag{21}$$

The orbital velocity for an orbit that is in the (r_1, ϕ) plane is

$$\dot{x}^2 + \dot{y}^2 = \dot{r}_1^2 + r_1^2\dot{\phi}^2 \tag{22}$$

$$\dot{y} = -\frac{b^2}{a^2}\frac{x}{y}\dot{x} \qquad \text{if } y \geq 0 \tag{23}$$

$$r_1\dot{\phi} = \frac{b^2}{a}\frac{1}{y}\dot{x} \qquad \text{if } y \geq 0 \tag{24}$$

Kepler's law is that the angular momentum

$$L = r_1^2\dot{\phi} \tag{25}$$

is constant. It does not follow from the equation of an ellpise. It follows from Euler-Lagrangian equations for a test mass m_1 circulating a spherically symmetric gravitational field created by a mass m_2 at *(−c, 0)*. The Lagrangean function for dynamic equations should normally be the sum of kinetic and potential energies

$$\mathcal{L} = E_k(t, q_i, \dot{q}_i) + E_p(t, q_i, \dot{q}_i) = E. \tag{26}$$

In order to find the dynamic equations, we minimize the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}dt = E(t_2 - t_1). \tag{27}$$

It is quite fine that the Lagrangean has a constant value like the total energy E, compare to Einstein's Lagrangean at (7). The Euler-Lagrange equations give the dynamic equations that keep the total energy at the constant value E. As an example, on the Earth surface the potential energy at the height s is $E_p = mgs$ and the kinetic energy is $E_k = (1/2)m\dot{s}^2$. We get the correct equation of motion from the Lagrangean

$$\mathcal{L} = E_k + E_p \tag{28}$$

$$\frac{\partial \mathcal{L}}{\partial s} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}} \tag{29}$$

$$mg = m\ddot{s}. \tag{30}$$

There is no sense in minimizing the integral over time of a function

$$\mathcal{L} = E_k(t, q_i, \dot{q}_i) - E_p(t, q_i, \dot{q}_i) = T - V \tag{31}$$

that does not have a lower bound. However, if we use radial coordinates, like (r_1, ϕ), then the acceleration is $-\ddot{r}_1$ because the r_1 vector points outside. Then we must write the Lagrangean as in (48), but it is only a question of the direction of r_1. Thus, in (r_1, ϕ), coordinates we write the Lagrangean as

$$\mathcal{L} = \frac{1}{2}m_1(\dot{r}_1^2 + r_1^2\dot{\phi}^2) - E_p(t, r_1, \phi). \tag{32}$$

Then Kepler's law

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m_1 \frac{d}{dt} r_1^2 \dot{\phi} = 0 \tag{33}$$

means that

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial \phi} E_p(t, r_1, \phi) = 0. \tag{34}$$

This is true only if m_1 is an insignificant test mass that does not disturb the field with its own field which is circulating on an elliptic orbit and for sure the position of m_1 depends on ϕ.

That is, Kepler's law is only approximatively true for planets orbiting the Sun. As Kepler's law is one of the postulates of Newtonian mechanics, it is difficult to understand why some people have thought that Newtonian mechanics should give an exact result for such a very small effect as the precession speed of Mercury and if it does not, then there would be needed a new theory like Einstein's geodesic Lagrangean.

Assuming that the potential energy is of the type

$$E_p = -GMm_1 \frac{1}{r_1} \tag{35}$$

Kepler's law holds, $r_1^2 \phi = L$ is constant and we can solve the Euler-Lagrange equation for r_1:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_1} = m_1 \ddot{r}_1$$

$$\frac{\partial \mathcal{L}}{\partial r_1} = m_1 r_1 \dot{\phi}^2 - GMm_1 \frac{1}{r_1^2}$$

$$\ddot{r}_1 = r_1 \dot{\phi}^2 - GM \frac{1}{r_1^2} \tag{36}$$

By using Kepler's law

$$\frac{d^2}{d\phi^2}\frac{1}{r_1} = \frac{d}{d\phi}\left(-\frac{\dot{r}_1}{\dot{\phi}}\frac{1}{r_1^2}\right) = -\frac{dt}{d\phi}\frac{d}{dt}\frac{\dot{r}_1}{L}$$

$$= -\frac{1}{\dot{\phi}L}\ddot{r}_1 = -\frac{r_1^2}{L^2}\ddot{r}_1 \tag{37}$$

and inserting to (53) gives an equation that r_1 in (40) fulfills

$$\frac{d^2}{d\phi^2}\frac{1}{r_1} + \frac{1}{r_1} = GM\frac{1}{L^2}. \tag{38}$$

Thus, the solution is an ellipse (40) and the angular momentum L is constant. In a gravitational field created by a point mass M the value of L is

$$L = \sqrt{GM}\frac{b}{\sqrt{a}} \tag{39}$$

if we assume that M is at the focal point (−c, 0).

The orbital period is calculated as

$$T = \int_0^T dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\dot{\phi}} d\phi = 2\int_{-a}^{a} \frac{1}{\dot{x}} dx = \frac{b^2}{aL} 2\int_{-a}^{a} \frac{r_1}{y} dx$$

$$= \frac{b^2}{aL} 2\frac{a}{b}\pi = 2\pi\sqrt{\frac{a^3}{Gm}} \tag{40}$$

for L as in (55).

If the mass M is not at the focal point, then the mass used in (55) is different and the orbital period (56) is also different. We could in principle find out where the Sun is related to the focal point by measuring the orbital period, but planets are so small compared to the Sun that this may be impossible in practice.

The exact position of the Sun is another issue that adds an error in the classical solution. We have placed the Sun at the focal point, but the Sun actually cannot be exactly at the focal point. We can see it by thinking of two equal masses m_1 and m_2 circulating each other's. By symmetry, the focal point must be at the center of mass. If m_1 is insignificant test mass and m_2 practically infinite, then m_2 is at the focal point. Between these two extreme situations the placement of the focal point must move continuously depending on the ratio of the masses. As the ratio of the mass of a planet and the Sun is not zero, the Sun cannot be exactly at the focal point.

It is also impossible that the focal point of the Sun-planet system is at the center of mass. If this were the case, then considering the two-body system Sun-Jupiter the Sun would be circulating the focal point with the orbital period of Jupiter. This means that every other planet that circulates the Sun would also have to circulate the same focal point and it would have to have the orbital period of Jupiter. This is not the case, planets have quite different orbital periods. Therefore the Sun must be much close to the focal point than it is in the coordinates where the focal point is the center of mass. The Sun must be so close to the focal point that the planets can have different orbital periods, yet the Sun cannot be exactly at the focal point.

This means that the movements of the planets are not quite separated, there is some small influence through the movement of the Sun. The Sun is in an orbit with some acceleration and if we choose a coordinate system (r_1, ϕ) where the Sun is at the focal point, then the origin of the coordinate system (r_1, ϕ) is accelerating and there are additional forces affecting the Sun.

The Sun, like Jupiter and Saturn, is not a solid mass, it is a gas ball and it compresses if a force is applied. In an accelerating orbit, or a coordinate system where the Sun is fixed but the coordinate system's origin is in accelerated orbit, there are acceleration forces. They do not need to do any work if the mass body is solid, but a gas ball compresses and the force makes work against forces that try to keep the mass body as spherical. The gas ball acts as a spring that is compressed by a force, it stores energy and at some other point it releases this energy. Therefore the total energy is not as in (26). There is additionally compressed energy. A small planet is reasonably solid and will not compress. As it mimics the movement of the Sun around the focal point but does not store energy by compression, it will have a mismatch between potential and kinetic energy: the sum of these energies is not constant; This mismatch is solved by precession of the orbit of the planet. This effect is outside Newtonian mechanics and requires understanding of how the Sun compresses under an acceleration force.

We see many small effects that can cause that Newtonian mechanics cannot give a precise result for the precession speed of Mercury. We now proceed to derive an equation for the precession speed.

Let us assume that the coordinates (r_1, ϕ) rotate around the focal point with angular velocity ω:

$$\phi = \phi_1 + \omega t$$

and we assume that the orbit is sufficiently close to an ellipse in (r_1, ϕ_1) coordinates, i.e.,

$$r_1 = a\frac{1-e^2}{1-e\cos(\phi_1)}.$$

We also assume the following conditions:

A1. All energy is in kinetic and potential energy, so no compression energy.

A2. The Sun is at the focal point of both planets we consider: Mercury and Jupiter.

A3. Kepler's law holds at the perihelion at $r_{1,\min}$ and aphelion at $r_{1,\max}$.

A4. The only effect causing precession of Mercury is that other planets change the gravitational force.

A3 means that the speed $v_{\phi_1,\max}$ at the perihelion relates to the speed of $v_{\phi_1,\min}$ at the aphelion as

$$v_{\phi_1,max} = \frac{r_{1,max}}{r_{1,min}} v_{\phi_1,min} \tag{41}$$

The speeds in the perihelion and aphelion in (r_1, ϕ) relate to the speeds in (r_1, ϕ_1) as

$$\begin{aligned} v_1 &= v_{\phi_1,max} + r_{1,min}\omega \\ v_2 &= v_{\phi_1,min} + r_{1,max}\omega \end{aligned} \tag{42}$$

Using (57) we get

$$v_1^2 - v_2^2 = \frac{r_{1,max}^2 - r_{1,min}^2}{r_{1,min}^2} v_{\phi_1,min}^2 - (r_{1,max}^2 - r_{1,min}^2)\omega^2 \tag{43}$$

$$= \frac{4ac}{(a-c)^2} v_{\phi_1,min}^2 - 4ac\omega^2. \tag{44}$$

The assumption that in (r_1, ϕ_1) the orbit is an ellipse means that at the perihelion and aphelion where *y = 0* we can calculate the centrifugal force as

$$\dot{x} = -\frac{a^2}{b^2}\frac{y}{x}\dot{y} \tag{45}$$

$$\ddot{x} = -\frac{a^2}{b^2}\frac{1}{x}\dot{y}^2 - \frac{a^2}{b^2} y \frac{d}{dt}\frac{\dot{y}}{x} \tag{46}$$

$$\ddot{x}|_{y=0} = \mp \frac{a^2}{b^2}\frac{1}{a}\dot{y}^2 \tag{47}$$

The absolute value of the centrifugal force at the perihelion is

$$F_{c,1} = m_1 \frac{a}{b^2} v_1^2 \tag{48}$$

and at the aphelion

$$F_{c,2} = m_1 \frac{a}{b^2} v_2^2$$

We assume that the gravitational force at the perihelion is

$$F_{g,1} = \alpha_1 G m_1 m_2 \frac{1}{r_{1,min}^2} \tag{49}$$

and at the aphelion

$$F_{g,2} = \alpha_2 G m_1 m_2 \frac{1}{r_{1,max}^2}$$

where α_1, α_2 describe the change of the gravitational force because of other planets. Thus,

$$v_1^2 - v_2^2 = G m_2 \frac{b^2}{a}\frac{\alpha_1}{(a-c)^2} - G m_2 \frac{b^2}{a}\frac{\alpha_2}{(a+c)^2} \tag{50}$$

$$= G m_2 \frac{1}{a}\frac{\alpha_1 (a+c)^2 - \alpha_2 (a-c)^2}{a^2 - c^2}. \tag{51}$$

From (61) and (62) comes

$$v_2 = \sqrt{Gm_2}\sqrt{\frac{a-c}{a+c}}\sqrt{\frac{\alpha_2}{a}}. \tag{52}$$

Equations (59) and (64) give two expressions for the left side of the equations. Inserting (65) gives after some manipulation a second order equation for ω

$$\omega^2 - 2\omega\frac{b}{4ac}\sqrt{Gm_2}\sqrt{\frac{\alpha_2}{a}} + \frac{b^2}{(4ac)^2}Gm_2\frac{\alpha_1-\alpha_2}{a} \tag{53}$$

The solution is

$$\omega = \frac{b}{4ac}\sqrt{Gm_2}\sqrt{\frac{\alpha_2}{a}}\left(1-\sqrt{2-\frac{\alpha_1}{\alpha_2}}\right) \tag{54}$$

The values to be inserted to (67) are: the gravitation constant $G=6.6743*10^{-11}m^3kg^{-1}s^{-2}$, the mass of the Sun $m^2=1.9891*10^{30}kg$, for Mercury: semi-major axis $a=5.7895*10^{-10}m$, $e=0.206$, $c=ea=1.1926*10^{-10}m$, semi-minor axis $b=\sqrt{a^2-c^2}=5.6653*10^{-10}m$, $r_{a,\max}=a+c$, $r_{a,\min}=a-c$. The measured precession of 5600 archseconds in a century is $\omega=8.6*10^{-12}s^{-1}$.

We can assume that α_1, α_2 are small and express them as $\alpha_i=1-\gamma_i$. In the first order

$$1-\sqrt{2-\frac{\alpha_1}{\alpha_2}} = \frac{1}{2}(\gamma_1-\gamma_2) \tag{55}$$

and the first order approximation for ω is

$$\omega = \frac{b}{8ac}\sqrt{Gm_2}\sqrt{\frac{1}{a}}(\gamma_1-\gamma_2). \tag{56}$$

Inserting numbers

$$\omega = 4.9*10^{-7}(\gamma_1-\gamma_2)\ s^{-1}. \tag{57}$$

We notice that (70) is very much what we would expect: $\gamma_1-\gamma_2$ should be $3.51*10^{-5}$ to give the measured value. Jupiter is about thousand times smaller than the Sun and its orbit is about ten times larger than that of Mercury, therefore the gravitational force from Jupiter to Mercury should be about $1/1000^*100$ of that of the Sun. This is just the 10^{-5} size. Let us make a very elementary estimation of the effect of Jupiter on Mercury's perihelion and aphelion. At the perihelion the gravitational field from Jupiter might be roughly

$$-Gm_J\frac{1}{a_J+r_{1,min}}$$

and at the aphelion roughly

$$-Gm_J\frac{1}{a_J-r_{1,max}}$$

where a_J is the semi-major axis of Jupiter, $a_j = 77.8473*10^{10}m$ and $m_J = 1.898*10^{27}kg$ is the mass of Jupiter. Then

$$\alpha_1(-Gm_2\frac{1}{r_{1,min}}) = -Gm_2\frac{1}{r_{1,min}} + Gm_J\frac{1}{a_J+r_{1,min}}$$

$$\alpha_1 = 1 - \frac{m_J}{}\frac{r_{1,min}}{a_J+r_{1,min}} \tag{58}$$

And

$$\alpha_2 = 1 - \frac{m_J}{m_2}\frac{r_{1,max}}{a_J-r_{1,max}}$$

$$\gamma_1-\gamma_2 = \frac{m_J}{m_2}\left(\frac{r_{1,min}}{a_J+r_{1,min}} - \frac{r_{1,max}}{a_J-r_{1,max}}.\right)$$

Notice that $\gamma_1-\gamma_2<0,$ so ω is negative, opposite to what we observe. We can ignore this issue because the example only demonstrates the strength of Jupiter's influence. We should put Jupiter and Mercury to different positions to get the direction of ω correct. Ignoring the sign, the strength is correct:

$$|\gamma_1-\gamma_2| = \frac{m_J}{m_2}\frac{2a^2-2c^2+2ca_J}{(a_J+c)^2-a^2} = 3.863*10^{-5}.$$

This gives the precession speed

$$\omega = 4.9*10^{-7}\frac{1}{2}3.863*10^{-5}\ s^{-1} = 9.46\ s^{-1}$$

which is not bad for such a simple approximation. Using better approximations 19th century astronomers managed to explain over 99% of the measured 5600 archseconds, mostly with the effect of the other planets.

Thus, tracking the positions of the other planets one can get quite good approximations for the measured precession speed of Mercury. The size of the measured ω is quite on the range of effects of planets, but here comes a caveat. The time series of Mercury's perihelions and aphelions is relatively short. Mercury is at the perihelion 415 times in a century and precise measurements have been made maybe for 500 years. There cannot be much more than some 2000 perihelion points in the record. Compare this to the presumed length of the precession cycle. With 5600 archseconds in a century one full cycle takes over 23,000 years. Nobody has ever measured a single full cycle. There is no good reason to assume that Mercury ever makes a full precession cycle. Instead, there is a reason to suspect that planet orbits only wobble and do not make full precession cycles: why else should the orbits of all planets be now pointing to roughly the same direction.

The question of what in reality is the precession speed of Mercury is not answered by experimental measurements. From measurements we only get the precession speed at this our time. In some thousand years the precession speed can be quite different. The planerary system has existed for billions of years. If there is a long term force, e.g., from Jupiter or other planets, that gives Mercury some small precession speed, then this speed continues in our times because of conservation of angular momentum, while we cannot see in our time any force that causes this precession. This may be the origin of the 43 archseconds.

Let us next calculate what is the long term effect of Jupiter on Mercury's precession. The result may be surprising: the long term effect is one hundred times smaller than the effect we see now. The forces that cause the effect now cancel each other's when the observation time is very long, but all forces do not cancel, there remains long term effects. The long term effect of Jupiter on Mercury's perihelionic precession is quite in the range of this missing 43 archseconds. We get roughly 54 archseconds in a century and should remember that this 43 archseconds is the minimum unexplained precession component: the unexplained part can be longer because in order to get this 43 archseconds every explanation has been pushed to its limits, to explain as much as possible. It is not likely that every mechanism should explain up to its maximum limits.

Long Term Effect of a Planet on the Precession Speed of Mercury

The average gravitational field at a place *(h, 0)* caused by a mass body *m* moving on an elliptic orbit with constant angular momentum is

$$\phi_{ave}(x_1, x_2) = -Gm\frac{1}{W(x_1, x_2)}\int_{x_1}^{x_2}\frac{r_1}{y}\frac{1}{s}dx \tag{59}$$

where r_1 is in (28), *y* and *x* in (27)

$$s = \sqrt{(x-h)^2 + y^2} = r_1\sqrt{1 - x\frac{2(c+h)}{r_1^2} + \frac{h^2 - c^2}{r_1^2}} \tag{60}$$

and the weight W is from the orbital time formula

$$W(x_1, x_2) = \int_{x_1}^{x_2}\frac{r_1}{y}dx. \tag{61}$$

The integral (71) can be calculated to the desired precision from a series expansion of (72). Let the numbers in (71) correspond to the orbit of Jupiter.

The semi-major axis $a = 77.8473 * 10^{10} m$, semi-minor axis $b = 77.7549 * 10^{10} m$, $c = 3.79116 * 10^{10} m, e = 0.0487$. The values of *h* that are of interest to us are $h_1 = -(c + r_{1,\min}), r_{1,\min} = 4.5969 * 10^{10} m$ *and* $h_2 = r_{1,\max} - c,\ h_2 = r_{1,\max} = 6.982 * 10^{10} m$ are the perihelion and aphelion values for Mercury. We assume that the orbits of Mercury and Jupiter have the same focal point and the Sun is at this point. The parameter *x* in (72) ranges from $-a$ to a. We can insert the numbers to (72) and notice that the term under the square root is

$$\sqrt{1+z} = 1 + \frac{1}{2}z + \frac{3}{8}z^2 + O(z^3) \tag{62}$$

and that $|z|$ is smaller than 0.18. This means that in order to get two significant numbers (error in 10^{-3}) we need a second order approximation in (72). This precision is sufficient for us.

Integral (71) can be calculated with the transform

$$\int \frac{dx}{yr_1^k} = -\frac{a^k}{b^{2k}} \int \frac{(ez+1)^{k-1}dz}{\sqrt{1-z^2}} \tag{63}$$

Where

$$z = -a\frac{x+c}{cx+a^2} \tag{64}$$

and either partially integrating or cancelling one term r_1 in the denominator. In the second order approximation we need

$$\int \frac{dx}{y} = \frac{a}{b} \arcsin\left(\frac{x}{a}\right) \tag{65}$$

$$\int_{-a}^{a} \frac{dx}{y} = \frac{a}{b}\pi$$

$$\int \frac{dx}{yr_1} = \frac{a}{b^2} \arcsin\left(a\frac{x+c}{cx+a^2}\right) \tag{66}$$

$$\int_{-a}^{a} \frac{dx}{yr_1} = \frac{a}{b^2}\pi$$

$$\int \frac{dx}{yr_1^2} = \frac{a^2}{b^3}\left(\left(x+\frac{a^2}{c}\right)^{-1}\sqrt{1-\frac{x^2}{a^2}} + \frac{1}{b}\arcsin\left(a\frac{x+c}{cx+a^2}\right)\right) \tag{67}$$

$$\int_{-a}^{a} \frac{dx}{yr_1^2} = \frac{a^2}{b^4}\pi$$

$$\int \frac{dx}{yr_1^3} = -\frac{a^3}{b^6}\left(-\left(1+\frac{1}{2}e^2\right)\arcsin(z) + \sqrt{1-z^2}\left(\frac{1}{2}e^2 z + 2e\right)\right) \tag{68}$$

$$\int_{-a}^{a} \frac{dx}{yr_1^3} = \frac{a^3}{b^6}\pi\left(1+\frac{1}{2}e^2\right)$$

$$\int \frac{dx}{yr_1^4} = \frac{a^4}{b^8}\left(-\left(1+3e^2\right)\arcsin(z) + \sqrt{1-z^2}\left(3e + \frac{3}{2}e^2 z + \frac{1}{3}e^3 z^2 + \frac{2}{3}e^3\right)\right) \tag{69}$$

$$\int_{-a}^{a} \frac{dx}{yr_1^4} = \frac{a^4}{b^8}\left(1+3e^2\right)\pi$$

$$\int \frac{Ax^2+Bx+C}{yr_1^2} = A\frac{a^2}{c^2}\int \frac{dx}{y} + \left(B\frac{a}{c} - A\frac{2a^2}{c^2}\right)\int \frac{dx}{yr_1}$$

$$+\left(A\frac{a^4}{c^2} - B\frac{a^2}{c} + C\right)\int \frac{dx}{yr_1^2} \tag{70}$$

$$\int_{-a}^{a} \frac{Ax^2+Bx+C}{yr_1^2} = A\pi\frac{a}{e^2 b}\left(1 - 2\frac{a}{b} + \frac{a^3}{b^3}\right)$$

$$+\left(B\pi\frac{a}{c} - A\frac{2a^3}{c^2}\right) + C\pi\frac{a^2}{b^4} \tag{71}$$

$$\int \frac{Ax^2+Bx+C}{yr_1^4} = A\frac{a^2}{c^2}\int \frac{dx}{yr_1^2} + \left(B\frac{a}{b} - A\frac{2a^2}{c^2}\right)\int \frac{dx}{yr_1^3}$$

$$+\left(A\frac{a^3}{c^2} - B\frac{a^2}{c} + C\right)\int \frac{dx}{yr_1^2} \tag{72}$$

$$\int_{-a}^{a} \frac{Ax^2+Bx+C}{yr_1^2} = A\pi\frac{a^2}{e^2 b^4}\left(1 - 2\frac{a^2}{b^2}\left(1+\frac{1}{2}e^2\right) + \frac{a^4}{b^4}\left(1+\frac{3}{2}e^2\right)\right)$$

$$+B\pi\frac{a^3}{eb^6}\left(1+\frac{1}{2}e^2 - \frac{a^2}{b^2}\left(1+\frac{3}{2}e^2\right)\right)$$

$$+C\pi\frac{a^4}{b^8}\left(1+\frac{3}{2}e^2\right) \tag{73}$$

The second order approximation is

$$\frac{1}{s}=\frac{1}{r_1}\left(1+(c+h)\frac{x}{r_1^2}+\frac{1}{2}(c^2-h^2)\frac{1}{r_1^2}+(c+h)^2\frac{x^2}{r_1^4}\right)$$

$$+\frac{1}{r_1}\left(-\frac{3}{4}(ch^2-hc^2+h^3-c^3)\frac{x}{r_1^4}+\frac{3}{8}(c^2-h^2)^2\frac{1}{r_1^4}\right). \quad (73)$$

Integrating gives

$$\int_{-a}^{a}\frac{r_1dx}{ys}$$

$$=\frac{a}{b}\pi-(c+h)\frac{a}{b^2}\pi\frac{e}{1-e^2}+\frac{1}{2}(c^2-h^2)\frac{a^2}{b^4}\pi$$

$$+\frac{3}{2}(c+h)^2\frac{a^2}{b^4}\pi\left(\frac{3}{2}\frac{a^4}{b^4}-\frac{a^2}{b^2}+\frac{e^2}{(1-e^2)^2}\right)$$

$$-\frac{3}{4}(ch^2-hc^2+h^3-c^3)\frac{a^3}{b^6}\pi e\left(-\frac{1}{1-e^2}+\frac{1}{2}-\frac{3}{2}\frac{a^2}{b^2}\right)$$

$$+\frac{3}{8}(h^2-c^2)^2\frac{a^4}{b^8}\pi\left(1+\frac{3}{2}e^2\right) \quad (73)$$

Derivating the second order approximation of the field with respect to h gives an approximation of the force, but notice that we have not yet divided by W, so the result is not yet force. We will drop terms with e^2 because the approximation has an error term of the size 10^{-10} and for Jupiter $e^2=2.3*10^{-3}$.

$$I=\frac{d}{dh}\int_{-a}^{a}\frac{r_1dx}{ys}$$

$$= -\frac{a}{b^2}\pi\frac{e}{1-e^2} - h\frac{a^2}{b^4}\pi$$

$$+3(c+h)\frac{a^2}{b^4}\pi\left(\frac{3}{2}\frac{a^4}{b^4} - \frac{a^2}{b^2}\right)$$

$$-\frac{3}{4}(2ch - c^2 + 3h^2)\frac{a^3}{b^6}\pi e\left(-\frac{1}{2} - \frac{3}{2}\frac{a^2}{b^2}\right)$$

$$+\frac{3}{2}h(h^2 - c^2)\frac{a^4}{b^8}\pi \tag{74}$$

Constant forces cancel when we calculate $\upsilon_1^2 - \upsilon_2^2$ (63). Therefore we drop them:

$$I = -h\frac{a^2}{b^4}\pi$$

$$+3h\frac{a^2}{b^4}\pi\left(\frac{3}{2}\frac{a^4}{b^4} - \frac{a^2}{b^2}\right)$$

$$-\frac{3}{4}(2ch + 3h^2)\frac{a^3}{b^6}\pi e\left(-\frac{1}{2} - \frac{3}{2}\frac{a^2}{b^2}\right)$$

$$+\frac{3}{2}h(h^2 - c^2)\frac{a^4}{b^8}\pi \tag{75}$$

We take the leading term of (87) as the other terms are clearly smaller:

$$I = h\frac{a^2}{b^4}\pi\left(-1 + 3\frac{a^2}{b^2}\left(\frac{3}{2}\frac{a^2}{b^2} - 1\right)\right)$$

$$= h\frac{a^2}{b^4}\pi\left(-1 + 3\frac{a^2}{b^2}\left(\frac{1}{2}\frac{a^2}{b^2} - \frac{-e^2}{1-e^2}\right)\right)$$

Dropping e^2 terms

$$= h\frac{a^2}{b^4}\pi\left(-1+\frac{3}{2}\frac{a^4}{b^4}\right)$$

Dividing with W and obtaining the force

$$W = \int_{-a}^{a}\frac{r_1 dx}{y} = \frac{a^2}{b}\pi$$

$$F = -Gm_1 m\frac{I}{W} = Gm_1 m_2\frac{h}{b^3}\left(1-\frac{3}{2}\frac{a^4}{b^4}\right)$$

Since

$$\frac{a^4}{b^4} = \left(\frac{1}{1-e^2}\right)^2 = 1+2e^2+O(e^4)$$

we simplify the force to

$$F = -Gm_1 m\frac{h}{2b^3}.$$

This force comes from potential of the type h^2, but that is because of the approximation that we used. The force has a fixed value at both values of *h* that we are interested in. We find a potential that is of the correct type

$$\psi = -Gm\frac{1}{r}$$

and gives the same force at h^1 and h^2. We now denote the values for Jupiter by an index. Thus, in (89) $b=b_J, m=m_J, c=c_J$ not to confuse them with the values for Mercury:

Thus, at $c_J+h=-r_{1,\min}$

$$\psi_1 = -Gm\frac{-r_{1,min}-c_J}{2b^3}r_{1,min} \tag{76}$$

and at $r_{1,\max}=c_J+h$

$$\psi_2 = -Gm\frac{r_{1,max} - c_J}{2b^3} r_{1,max} \tag{77}$$

Then we still have to get α_i as in (70) and γ_i.

$$\gamma_1 = \frac{m_J}{m_2} \frac{(-r_{1,min} - c_J) r^2_{1,min}}{2b^3_J}$$

$$\gamma_2 = \frac{m_J}{m_2} \frac{(r_{1,ax} - c_J) r^2_{1,min}}{2b^3_J}$$

Now we can estimate the size of the long term effect of Jupiter on the periheliotic precession of Mercury:

$$\gamma_1 - \gamma_2 = -\frac{m_J}{m_2} \frac{-r^3_{1,min} - c_J r^2_{1,min} - r^3_{1,max} + c_J r^2_{1,max}}{2b^3_J}$$

$$= \frac{m_J}{m_2} \frac{a^3 \left(1 + 3e^2 - 2e\frac{c_J}{a}\right)}{b^3_J}$$

Inserting numbers $\frac{m_J}{m_2} = 0.9542 * 10^{-3}, \frac{a^3}{b_J^3} = 4.127 * 10^{-4}\ and\ 1 + 3e^2 - 2e\frac{cJ}{a} = 0.8575.$ The result is $\gamma_1 - \gamma_2 = 3.3625 * 10^{-7}$ and $\omega = 4.9 * 10^{-7} * 1.813 * 10^{-7} s^{-1} = 16.5 * 10^{-14}$ which is about 107 archseconds per century.

Jupiter's year is about 12 years, so the planet is at each place in its orbit every 12th year, but in the calculation we also assume that Mercury is at its perihelion and that this perihelion is in a particular place with Mercury's orbit pointing to the same direction as that of Jupiter. This assumption is not fully valid even today and when Mercury precesses more, this assumption cannot hold. Let us take half of 107 archseconds per century as a rough estimate to account for the angle between the semi-major axes of Mercury's orbit and Jupiter's orbit. Thus, the predicted precession is about 54 archseconds in a century.

This is my proposal for an unknown mechanism that can cause precession of Mercury's perihelion. There must exist some unknown or ignored mechanism that explains the 43 archseconds, and probably a bit more. Einstein's explanation cannot be correct. It is difficult to find some mechanisms that has not been considered, but there are very long time effects, all forces do not cancel even in a very long time. The solar system has had billions of years' time and such long time effects have been compensated by precession because the energy budget must hold. If such a long term calculation shows that there is an energy inbalance, it must result to something that fixes it, like to very small precession. In a relatively short observation period, like some hundred years, we cannot see these long term mechanisms. The short term mechanisms are much stronger because forces do not cancel. Constant potential terms that come out of the integration in (85) do not mean constant potential in anything than in the average. At each time moment the potential that is shown as not dependent on h in (85) is a potential that has a clear gradient pointing to Jupiter. This is why there appears these hundred times larger forces than in the average.

A Serious Error in Einstein's Formula for the Precession

Einstein's calculation, or a form of it that seems to be used today for teaching students, can be found from Owen Biesel's paper [6]. The paper derives the Schwarzschild metric, but let as start from the point where the geodesic Lagrangean appears to the calculations

$$\mathcal{L} = -\left(1-\frac{R_s}{r}\right)\dot{T}^2 + \left(1-\frac{R_s}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2. \tag{78}$$

[6] says that $L = -1$. If this is so, then he can use this Lagrangean instead of

$$\mathcal{L} = \sqrt{\left(1-\frac{R_s}{r}\right)\dot{T}^2 - \left(1-\frac{R_s}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2}. \tag{79}$$

as the square root term $(2L)^{-1}$ *is* 2^{-1}. Let us assume $L = -1$ and calculate like [6]. notices that

$$\frac{\partial\mathcal{L}}{\partial T} = \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{80}$$

Therefore the Euler-Lagrange equations for T and ϕ give

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{T}} = \frac{d}{d\tau}2\left(1-\frac{R_s}{r}\right)\dot{T} = 0$$

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{d}{d\tau}2r^2\phi = 0$$

and [6] gets Kepler's law and the energy conservation law:

$$L = r^2\dot{\phi}$$

$$E = \left(1-\frac{R_s}{r}\right)\dot{T} \tag{81}$$

are constants. Then the paper again uses the assumption that $L = -1$ inserting (94) to L and solving $\dot{r}^2$

$$\dot{r}^2 = E^2 - 1 + \frac{R_s}{r} - \frac{L^2}{r^2} + \frac{R_s L^2}{r^3} \tag{82}$$

Writing

$$\dot{r} = \frac{dr}{d\tau} = \frac{d\phi}{d\tau}\frac{dr}{d\phi} = \dot{\phi}r' = \frac{L}{r^2}r'$$

[6] gets (95) to the form

$$(r')^2 = \frac{E^2-1}{L^2}r^4 + \frac{R_s}{L^2}r^3 - r^2 + R_s r \tag{83}$$

There are four points when $r'=0,$ two of them being $R_+ = a+c$ *and* $R = a-c,$ the aphelion and perihelion points of Mercury. Here $c = ea = 1.1926*10^{10}\,m$ and not the speed of light. One root is $r = 0$ and the fourth root [6] denotes by ε , but let us denote it by R_4 just to remind that it is meters. Thus

$$\frac{E^2-1}{L^2}r^4 + \frac{R_s}{L^2}r^3 - r^2 + R_s r = \frac{1-E^2}{L^2}r(R_+ - r)(r-R_-)(r-R_4). \tag{84}$$

[6] solves E^2 and L^2 using the two roots $R_+ = a + c$ and $R_- = a - c$. Then the paper calculates an integral

$$\int_{R_-}^{R_+} \frac{dr}{\sqrt{r(R_+ - r)(r-R_-)(r-R_4)}}$$
$$= \int_{R_-}^{R_+} \frac{dr}{r\sqrt{(R_+ - r)(r-R_-)(1-R_4 r^{-1})}}.$$

A first order approximation is made

$$\frac{1}{\sqrt{1-\frac{R_4}{r}}} = 1 + \frac{1}{2}\frac{R_4}{r} + \text{error term.}$$

Einstein's precession speed formula comes from the integral

$$I = \frac{R_4}{2}\int_{R_-}^{R_+} \frac{dr}{r^2\sqrt{(R_+ - r)(r-R_-)}}$$

The integral gives

$$I = \frac{1}{\sqrt{R_+R_-}}\frac{\pi R_4}{4D}$$

where

$$D = \frac{R_+R_-}{R_+ + R_-} = \frac{b^2}{2a}.$$

We used here $R_{+}+R_{-}=(a+c)+(a-c)=2a \; and \; R_{+}R_{-}=a^{2}-c^{2}=b^{2}$.

Inserting D we get

$$I=\frac{1}{b}\frac{\pi R_{4}2a}{4b^{2}}=\frac{R_{4}}{2}\frac{a}{b^{3}}\pi. \tag{85}$$

[6] notices that

$$\frac{L^{2}}{1-E^{2}}=\frac{R_{+}R_{-1}}{1-\frac{R_{s}}{D}}$$

and taking the constant from the polynomial (97) shows that

$$R_{4}=\frac{R_{s}}{1-\frac{R_{s}}{D}}.$$

The final result that gives the exact 43 missing arch seconds is

$$\phi_{+}-\phi_{-}=\int_{R_{-}}^{R_{+}}\frac{dr}{r'}=\sqrt{\frac{L^{2}}{1-E^{2}}}\int_{R_{-}}^{R_{+}}\frac{dr}{\sqrt{r(R_{+}-r)(r-R_{-})(r-R_{4})}}$$

Using the approximation the result is

$$\phi_{+}-\phi_{-}=\frac{1}{\sqrt{1-\frac{R_{s}}{D}}}\left(\pi+\frac{1}{\sqrt{R_{+}R_{-}}}\frac{R_{4}}{2}\int_{R_{-}}^{R_{+}}\frac{dr}{r^{2}\sqrt{(R_{+}-r)(r-R_{-})}}\right) \tag{86}$$

which gives Einstein's formula

$$\phi_{+}-\phi_{-}=\frac{\pi}{\sqrt{1-\frac{R_{s}}{D}}}\left(1+4\frac{R_{s}}{1-\frac{R_{s}}{D}}\right). \tag{87}$$

Thus, this formula does come from the Lagrangean, but it does not help. There is a serious error in the assumption that $L=-1$. Let us assume it is so and calculate the Euler-Lagrange equation for r that [6] did not do. Thus,

$$\frac{\partial\mathcal{L}}{\partial r}=\frac{1}{r^{2}\left(1-\frac{R_{s}}{r}\right)}\left(-R_{s}E^{2}-R_{s}\dot{r}^{2}+\frac{2L^{2}}{r}\left(1-\frac{R_{s}}{r}\right)^{2}\right) \tag{88}$$

And

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{r}}=\frac{1}{r^{2}\left(1-\frac{R_{s}}{r}\right)}\left(-2R_{s}\dot{r}^{2}+2\ddot{r}\left(1-\frac{R_{s}}{r}\right)\right). \tag{89}$$

From (103) we get

$$\ddot{r} = \frac{R_s}{2r(r-R_s)}\left(-E^2+\dot{r}^2\right) - \frac{L^2}{r^3} + \frac{L^2 R_s}{r^4} \tag{90}$$

Inserting $\dot{r}^2$ from (95) to (104)

$$\ddot{r} = \frac{R_s}{2r(r-R_s)}\left(-1+\frac{1}{r}\left(R_s - \frac{2L^2}{R_s} + 2L^2\right) + \frac{L^2}{r^2}\right) \tag{91}$$

We get another equation for $\ddot{r}$ by derivating (95) with respect to τ

$$2\dot{r}\ddot{r} = -\frac{R_s}{r^2}\dot{r} + 2\frac{L^2}{r^3}\dot{r} - 3\frac{R_s L^2}{r^4}\dot{r}$$

$$\ddot{r} = -\frac{R_s}{2r^2} + \frac{L^2}{r^3} - \frac{3}{2}\frac{R_s L^2}{r^4} \tag{92}$$

If $\mathcal{L} = -1$, then

$$-1+\frac{1}{r}\left(R_s - \frac{2L^2}{R_s} + 2L^2\right) + \frac{L^2}{r^2}$$

equals

$$= \frac{2r(r-R_s)}{R_s}\left(-\frac{R_s}{2r^2} + \frac{L^2}{r^3} - \frac{3}{2}\frac{R_s L^2}{r^4}\right)$$

$$= -1+\frac{1}{r}\left(R_s + \frac{2L^2}{R_s}\right) - \frac{L^2}{r^2} + 3\frac{R_s L^2}{r^3}$$

We see that they are not equal. The assumption L = −1 is wrong. L is not constant and therefore the Euler-Lagrange equations for this geodesic Lagrangean are wrong. For a correct calculation of geodesics in the Schwarzschild metric, see [7-9]. The geodesic equations have long and difficult expressions.

Let us still investigate what is the curve that Einstein's geometric Lagrangean gives. It is not a rotating ellipse. A rotating ellipse has the formula

$$r = \frac{a(1-e^2)}{1-e\cos(\phi-\omega t)} \tag{93}$$

Assuming that ω is small, the orbital time when Mercury is circling the Sun is closely approximated by and L is closely approximated by

$$T = 2\pi\sqrt{\frac{a^2}{GM}} \tag{94}$$

and L is closely approximated by

$$L = \sqrt{\frac{GM}{a}}b \tag{95}$$

Derivating *r* we get

$$r' = -\frac{e}{a(1-e^2)}\sin(\phi - \omega t)\left(1 - \frac{\omega}{\dot{\phi}}\right)r^2.$$

There are two zeros in the range $0 \leq \phi \leq \pi$. They are the zeros of sin $(\phi - \omega t)$ and they are

$$\phi_- = 0 \qquad \phi_+ = \pi + \omega\frac{T}{2}.$$

The other zeros are not possible: $r = 0$ does not happen on the orbit of the ellipse and $1 - \omega(\dot{\phi})^{-1} = 0$ does not happen when ω is small.

Eliminating sin $(\phi - \omega t)$ by using the following equation derived from (107)

$$\cos(\phi - \omega t) = \frac{1}{e}\left(1 - \frac{a(1-e^2)}{r}\right)$$

we get after some manipulation

$$r'^2 = \frac{r^2}{a^2(1-e^2)}(R_+ - r)(r - R_-)\left(1 - \frac{\omega}{\dot{\phi}}\right)^2$$

Where $R_+ = a + c = c = a(1+e), R_- = a - c = a(1-e).$ Thus

$$r' = \frac{1}{a\sqrt{1-e^2}}r\sqrt{(R_+ - r)(r - R_-)}\left(1 - \frac{\omega}{\dot{\phi}}\right)$$

Noticing that $a\sqrt{1-e^2} = b$

$$r' = \frac{1}{b}r\sqrt{(R_+ - r)(r - R_-)}\left(1 - \frac{\omega}{\dot{\phi}}\right) \tag{96}$$

This is an equation of a rotating ellipse.

Einstein has in (96)-(97)

$$r' = \sqrt{\frac{1-E^2}{L^2}}r\sqrt{(R_+ - r)(r - R_-)}\sqrt{1 - \frac{R_4}{r}}.$$

Noticing that

$$\sqrt{\frac{1-E^2}{L^2}} = \frac{\sqrt{1 - \frac{R_s}{D}}}{b} = \frac{1}{b}\sqrt{1 - \frac{2aR_s}{b^2}}$$

and that $R_4 = R_s/(1-R_s/D)$ is very close to $R_s << r$ we can approximate

$$r'^2 = \frac{1}{b} r\sqrt{(R_+ - r)(r - R_-)}\sqrt{1-\frac{2aR_s}{b^2}}\sqrt{1-\frac{R_4}{r}}. \tag{97}$$

$$= \frac{1}{b} r\sqrt{(R_+ - r)(r - R_-)}\left(1 - R_s\left(\frac{a}{b^2}+\frac{1}{2r}\right)\right) + \text{error term} \tag{98}$$

Notice that (111) is not an equation of a rotating ellipse and that the approximation (112) comparing it to (110) gives a first order approximation

$$\frac{\omega}{\dot\phi} = R_s\left(\frac{a}{b^2}+\frac{1}{2r}\right) \tag{99}$$

which is totally impossible because though Kepler's law

$$L = r^2\dot\phi \tag{100}$$

need not hold exactly, it certainly is a very good approximation in the rotating coordinates (r,ϕ).

We calculate as in [6] eliminating $\dot\phi$ in (110) by (114)

$$\int_{\phi_-}^{\phi_+} d\phi = \int_{R_-}^{R_+} \frac{1}{\frac{dr}{d\phi}} dr = \int_{R_-}^{R_+} \frac{dr}{r'}$$

$$= b\int_{R_-}^{R_+} \frac{dr}{r\sqrt{(R_+ - r)(r - R_-)}\left(1-\frac{\omega}{\dot\phi}\right)}$$

and notice that the insertation $x = (a/c)(r-a), r_1 = r,$ changes

$$\int_{R_-}^{R_+} \frac{dr}{r^\alpha\sqrt{(R_+ - r)(r - R_-)}} = \frac{a}{b}\int_{-a}^{a}\frac{dx}{yr_1^\alpha}$$

for any α Taking a first order approximation

$$\left(1-\frac{\omega r^2}{L}\right)^{-1} = 1 + \frac{\omega r^2}{L} + \text{error term}$$

we get a good approximation

$$\phi_+ - \phi_- = b\int_{-a}^{a} \frac{dx}{yr_1} + b\frac{\omega}{L}\int_{-a}^{a} \frac{r_1 dx}{y}.$$

We give some more formulas:

$$\int \frac{r_1 dx}{y} = -\frac{ca}{b}\sqrt{1-\frac{x^2}{a^2}} + \frac{a^2}{b}\arcsin\frac{x}{a}$$

$$\int_{-a}^{a} \frac{r_1 dx}{y} = \frac{a^2}{b}\pi$$

$$\int \frac{r_1^2 dx}{y} = -\frac{c}{b}\left(2a^2 + \frac{1}{2}cx)\right)\sqrt{1-\frac{x^2}{a^2}} + \frac{a^3}{b}\left(1+\frac{1}{2}\frac{c^2}{a^2}\right)\arcsin\frac{x}{a}$$

$$\int_{-a}^{a} \frac{r_1^2 dx}{y} = \frac{a^3}{b}\pi\left(1+\frac{1}{2}e^2\right)$$

The second formula we do not need here, but it is nice to know. We get

$$\phi_+ - \phi_- = b\left(\frac{\pi}{b} + \frac{\omega}{L}ab\pi\right)$$

and inserting L from (109)

$$\phi_+ - \phi_- = \pi + \pi\sqrt{\frac{a^3}{GM}}\omega.$$

The result is

$$\omega = \frac{\phi_+ - \phi_- - \pi}{\pi\sqrt{\frac{a^3}{GM}}} = \frac{\phi_+ - \phi_- - \pi}{\frac{1}{2}T}$$

as it should be, showing that the errors in the approximations cancel nicely. In order to reject Einstein's formula, it is enough to compare (110) and (111). Whatever (111) is, it is not what it should be: a rotating ellipse. It gives an impossible result (113).

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